XXXVI Olimpiada Iberoamericana de Matemáticas

Day 1

October 19, 2021

Problem 1. Let $P = \{p_1, p_2, \ldots, p_{10}\}$ be a set of 10 different prime numbers and let A be the set of all integers greater than 1 such that their prime factorizations contain only primes in P. Each element in A is colored in the following way:

a) each element in P has a distinct color,

b) if $m, n \in A$, then mn has the same color as m or n,

c) for each pair of distinct colors $\mathcal{R}$ and $\mathcal{S}$, there are no $j, k, m, n \in A$ (not necessarily distinct), with j, k colored $\mathcal{R}$ and m, n colored $\mathcal{S}$, such that both j divides m and n divides k.

Show that there is some prime in P such that all of its multiples in A have the same color.

Problem 2. Consider an acute triangle ABC, with $AC > AB$, and let Γ be its circumcircle. Let E and F be the midpoints of the sides AC and AB, respectively. The circumcircle of triangle CEF intersects Γ at X and C, with $X \neq C$. The line BX and the line tangent to Γ at A intersect at Y. Let P be the point on segment AB such that $YP = YA$, with $P \neq A$, and let Q be the point where AB intersects the line parallel to BC passing through Y. Show that F is the midpoint of PQ.

Note: The circumcircle of a triangle is the circle passing through its three vertices.

Problem 3. Let $a_1, a_2, a_3, \ldots$ be a sequence of positive integers and let $b_1, b_2, b_3, \ldots$ be the sequence of real numbers given by

$$b_n = \frac{a_1 a_2 \cdots a_n}{a_1 + a_2 + \cdots + a_n}, \text{ for } n \geq 1.$$

Show that if among every one million consecutive terms of the sequence $b_1, b_2, b_3, \ldots$ there is at least one integer, then there is some k such that $b_k > 2021^{2021}$.

Time: 4 hours and 30 minutes
Each problem is worth 7 points